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K-theory and S-duality: starting over from square 3

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ABSTRACT: Recently Maldacena, Moore, and Seiberg (MMS) have proposed a physical interpretation of the Atiyah-Hirzebruch spectral sequence, which roughly computes the K-homology groups that classify D-branes. We note that in IIB string theory, this approach can be generalized to include NS charged objects and conjecture an S-duality covariant, nonlinear extension of the spectral sequence. We then compute the contribution of the MMS double-instanton configuration to the derivation d_5 and as an application compute the (p, q)-string spectrum on the group manifold SU(3). We conclude with an M-theoretic generalization reminiscent of 11-dimensional E_8 gauge theory.

KEYWORDS: String Duality, D-branes, Tachyon Condensation.

Contents

1. Introduction

Various manifestations of K-theory appear to classify D-brane configurations at large distances and weak coupling [1, 2]. This is a consequence of the proposal [3] that each configuration is related, by the dynamical process of tachyon condensation, to a universe filled with non-BPS D9-branes [4] or D9-$\overline{\text{D9}}$ pairs whose tachyon fields are sections of bundles over the 10-dimensional spacetime. One can then define the configurations before and after tachyon condensation to be equivalent. Tachyon configurations are classified by K-theory [2] and so we learn that K-theory classifies D-brane configurations as well.

In IIB string theory, S-duality would suggest a similar classification of NS and RR charged objects. However, tachyon configurations on space filling branes do not obviously yield NS5-branes or fundamental strings.[1] Of course, one could simply impose S-duality and then arrive at a "K-theory" classification that includes NS charged objects, but some authors [6] have suggested that perhaps S-duality itself fails when discrete torsions are involved.

While the tachyon condensation approach to D-brane classification is difficult to generalize to other branes, the approach recently proposed by Maldacena, Moore, and Seiberg [7] generalizes beautifully. Consider all of the consistent time independent states and identify the ones related by physical processes, which are manifested in examples as "instantonic" branes. In the case of D-branes in an H-field, this prescription may be used to construct an Atiyah-Hirzebruch [8] spectral sequence which computes the K-group of a space (modulo an extension problem related to the dimension of torsional brane charges [9]). We introduce an extension of this procedure which includes fundamental strings, NS5-branes and branes in M-theory.

We will require three tools. The non-torsion part of the problem will be understood using the classical equations of motion and Bianchi identities from IIB and 11-dimensional supergravity. In particular, the nonlinearities in supergravity translate into corresponding nonlinearities in the "differentials" of the sequence. Incorporating torsion corrections in IIB will require a variation of the Freed-Witten anomaly, which restricts NS5-branes to wrap spin^c submanifolds of spacetime to avoid D-string worldsheet anomalies. The nontorsion classification in M-theory will be understood via an interpretation of M5-branes as defects in an E_8 bundle over the 11-dimensional spacetime [10], the restriction to 11 dimensions of a 12-dimensional construction that appeared in refs. [11, 12, 13].

In section 2, after briefly reviewing the Atiyah-Hirzebruch spectral sequence, we present our conjecture for an S-duality covariant extension of the sequence. Then in section 3 we review the facts that we will need from classical IIB supergravity and review branes ending on branes that wrap a cycle supporting flux. In the following section we present the MMS interpretation of the spectral sequence and generalize it to include NS5-branes and strings in the supergravity limit, where torsion is neglected. In the next section the Freed-Witten anomaly is reviewed and generalized, which allows torsion to be incorporated in the new sequence. In section 6 examples are given and in particular we consider the SU(3) WZW model in some detail. The double-instanton found in ref. [7] is generalized and its

[1] For an exception, see [5].

contribution to the differential d_5 is computed. This analysis is extended to include NS5-branes and strings in the SU(3) case. In the last section the techniques of this paper are applied to M theory.

2. The conjecture

2.1 The AHSS

The Atiyah-Hirzebruch spectral sequence [8] is an algorithm which relates cohomology and the K-groups $K^*(M)$. First introduce a filtration on $K(M)$ by defining

$$K_p = \text{Ker } K(M) \to K(M^p) \tag{2.1}$$

where M_p is the p-skeleton of M. The spectral sequence then computes, for example, the associated graded algebra of $K^1(M)$

$$\text{Gr}K^1 = \oplus_q K^1_{q+1}/K^1_q\,. \tag{2.2}$$

This process procedes through a series of approximations $K^1 \sim E_n$ which terminate after a finite number of iterations. The first approximation is integer-valued cohomology

$$E_1 = \bigoplus_{j \text{ odd}} E_1^j = \bigoplus_{j \text{ odd}} H^j(M,\mathbb{Z}) \qquad E_1' = \bigoplus_{j \text{ even}} E_1^j = \bigoplus_{j \text{ even}} H^j(M,\mathbb{Z})\,. \tag{2.3}$$

Successive approximations result from taking the cohomology of (2.3) with respect to a sequence of differentials

$$d_{p+2} : E_p^q \to E_p^{q+p+2} \qquad d'_{p+2} : E_p^{q-p-2} \to E_p^q \tag{2.4}$$

where p is odd. That is,

$$E_{p+2} = \ker(d_{p+2})/\text{Im}(d'_{p+2}) \tag{2.5}$$

giving E_{p+2} as an equivalence class of subsets of E_p. For sufficiently high p,

$$\text{Gr}K^1 = E_{p+2} \tag{2.6}$$

and then K^1 can be computed by the solution of an extension problem.

In IIA string theory only d_3 is needed, that is, the associated graded algebra is simply E_3. In IIB this is also true with the exception of d_5 acting on the 3 form fieldstrengths of 5-branes in the presence of a nontrivial H flux, which is sometimes nontrivial [7]. As we will review later,

$$d_3 = Sq^3 + H \tag{2.7}$$

and so

$$\text{Gr}K^1 = \text{Ker } (Sq^3 + H)/\text{Im}(Sq^3 + H) \tag{2.8}$$

up to d_5 corrections. Here K^1 is a twisted K-group and the twist is given by the NS fieldstrength H. Eq. (2.8) classifies which integral cohomology classes may be realized as RR fluxes in string theory. Our goal is to extend this formula to include NS fluxes in IIB.

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2.2 S-duality covariant AHSS

Following the suggestion of ref. [14], we propose the following modified Atiyah-Hirzebruch spectral sequence as a starting point for an S-duality covariant classification for fluxes in IIB string theory. Instead of beginning with the complex of odd dimensional cohomology classes (2.3), one begins with

$$E_1 = H^1 \bigoplus H^3 \bigoplus H^3 \bigoplus H^5 \bigoplus H^7 \bigoplus H^7 \,. \tag{2.9}$$

As we will explain below, the MMS interpretation of the differentials combined with a generalization of the Freed-Witten anomaly argument suggests

$$d_3^1(G_1) = (Sq^3 + H\cup)G_1 \tag{2.10}$$
$$d_3^3(G_3, H) = Sq^3(G_3 + H) + G_3 \cup H \tag{2.11}$$
$$d_3^{5a}(G_5) = (Sq^3 + H\cup)G_5 \qquad d_3^{5b}(G_5) = (Sq^3 + G_3\cup)G_5 \tag{2.12}$$
$$d_3^7(*G_3, *H) = Sq^3(*G_3 + *H) + H \cup *G_3 \,. \tag{2.13}$$

We will neither attempt to find formulas for d', nor will we claim that such formulas are nontrivial for the classification of fluxes. Later, when we classify charges, the choice of d' will be apparent from the physics.

Notice that the Sq^3 terms are trivial in all but (2.11). We claim that flux configurations which are not annihilated by the above differentials are anomalous. In particular, in the supergravity limit where one ignores the torsion terms, this condition is equivalent to the enforcement of supergravity equations of motion, to be reviewed in Subsection 3.1.

In the last equation $*G_3$ and $*H$ are to be treated as 7-forms which are independent of G_3 and H, as they yield independent cohomology classes. The confusing notation is meant to suggest that were one not only interested in topology, then one would find that the de Rham representations of the torsion parts of the 3 and 7-forms are related by the Hodge star. In this paper we will be concerned only with topology.

In contrast with the AHSS, the action of d_3 in eq. (2.11) is not always linear! This nonlinearity can be traced to the nonlinearity of the corresponding supergravity equations of motion. Notice that in the absence of the H-field and NS5 branes, the supergravity equations of motion, and hence the differential, becomes linear. So what do we mean by taking the cohomology with respect to d_3? As we will see when we review interpretation of this sequence due to Maldacena, Moore, and Seiberg (MMS), we mean simply that physical flux configurations are those annihilated by d_3. One physical consequence of this nonlinearity is the fact that the set of allowed fluxes need not form a group [15]. In the linear case we quotient by the image of d_3 because the corresponding states may decay via physical process. In fact there are many interesting examples where d_3 is linear, such as all of the examples studied by MMS. A weak form of this conjecture may be stated which only includes this subset of configurations. Of course when d_3 is nonlinear it is an abuse of language to continue to use to the phrases “differential operator” and “spectral sequence”.

We know that this is not a complete list of differential operators. In particular there are examples [7] that illustrate that anomaly cancellation can require the operator d_5, and new instantons can allow states to decay which are exact under d_5 but not under d_3. We consider some such configurations in detail in section 6. We leave the completion of this list and its interpretation to a sequel.

3. Review of branes in IIB supergravity

3.1 Classical equations of motion

The low energy effective theory of IIB string theory is 10-dimensional IIB supergravity, whose action contains the following terms (here the self-duality constraint for G_5 is imposed by hand):

$$\begin{aligned} S &\supset -\frac{1}{4\kappa_{10}^2}\int d^{10}x(-g)^{1/2}[\mathcal{L}_{NS}+\mathcal{L}_R+\mathcal{L}_{CS}] && (3.1)\\ \mathcal{L}_{NS} &= e^{-2\Phi}\left(-2R-8\partial_\mu\Phi\partial^\mu\Phi+|H|^2\right) && (3.2)\\ \mathcal{L}_R &= |G_1|^2+|G_3|^2+\frac{1}{2}|G_5|^2 && (3.3)\\ \mathcal{L}_{CS} &= H\wedge C_4\wedge dC_2 && (3.4)\end{aligned}$$

where we have defined the gauge invariant fieldstrengths

$$G_3 = dC_2 + H\wedge C_0 \qquad G_5 = dC_4+\frac{1}{2}H\wedge C_2-\frac{1}{2}B\wedge dC_2\,. \tag{3.5}$$

In terms of these fieldstrengths the Bianchi identities can be rewritten

$$ddB = dH = 0 \qquad ddC_{p-1} = d(G_p - H\wedge C_{p-3}) = dG_p - H\wedge G_{p-2} = 0\,. \tag{3.6}$$

The equations of motion are

$$d*H\sim G_3\wedge G_5 \qquad d*dC_{p-1} = d*G_p - H\wedge *G_{p+2} = 0\,. \tag{3.7}$$

We will be interested in Dp-branes, fundamental strings, NS5-branes, M2-branes and M5-branes, which couple to the connections C_{p+1}, B, $B^{\rm dual}$, C_3 and $C^{\rm dual}$ respectively. The corresponding fieldstrengths are

$$\begin{aligned} G_{p+2} &= dC_{p+1} - H\wedge C_{p-1} \qquad H = dB \qquad *H = dB^{\rm dual} && (3.8)\\ G_4 &= dC_3 \qquad *G_4 = d(C^{\rm dual})\,. && (3.9)\end{aligned}$$

3.2 Branes ending on branes

Imagine that a Dp-brane wraps a 3-cycle that supports k units of H flux. As a Dp-brane gives rise to a transverse G_{8-p} flux, $G_{8-p}\wedge H\neq 0$ and the supergravity equation of motion from subsection 3.1

$$H\wedge G_{8-p} = dG_{10-p} \tag{3.10}$$

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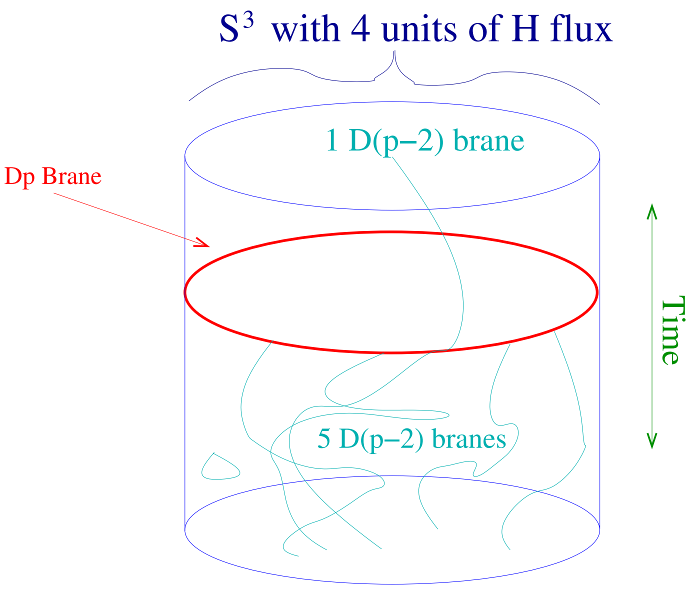

Figure 1: A Dp-brane wraps a 3-sphere that supports 4 units of H flux. Global worldsheet anomaly cancellation requires that 4 D$(p-2)$-branes end on this Dp-brane. Thus the lone Dp-brane is not allowed. Also the number of D$(p-2)$-branes is only defined modulo 4 because a dynamical process involving similarly wrapped Dp-brane instantons can create or destroy 4 of them at a time.

implies the existence of nonzero D$(p-2)$-brane current G_{10-p} emanating from the Dp-brane, which may be taken to be transverse to both the 3-cycle supporting the H flux and any S^{8-p} linking the Dp-brane. Integrating both sides one finds that the current can be associated with the presence of k units of D$(p-2)$-brane charge. Thus, k D$(p-2)$-branes must end on any Dp-brane wrapping a 3-cycle with k units of flux.

The above arguments can be used to demonstrate that a Dp-brane wrapped on a p-cycle with k units of G_p flux must be the endpoint of k fundamental strings. For the case $p=3$ this is the S-dual of k D-strings ending on a D3-brane wrapping a three-cycle with k units of H flux. For the case $p=5$ it is the S-dual of D-strings ending on an NS5-brane, a configuration to which we will now turn our attention.

Consider an NS5-brane that wraps a p-cycle Z supporting k units of G_p flux. Employing the S-dual of the above argument, the NS5 brane is linked by a 3-sphere such that

$$\int_{S^3} H = 1\,. \tag{3.11}$$

Now we may integrate the supergravity equation of motion

$$H \wedge G_p = dG_{p+2} \tag{3.12}$$

over $S^3 \times Z$ to see that there are k D$(6-p)$-branes intersecting every 3-sphere linking the NS5-brane and so k D$(6-p)$-branes must be threaded down the NS5-brane's throat.

4. Maldacena-Moore-Seiberg construction of the AHSS

4.1 Lifting cohomology to K-theory

The K-theory classification of D-branes in IIB string theory represents all configurations of D-branes as defects in the tachyon field on the worldvolume theory of a collection of unstable D9-$\overline{\text{D9}}$ pairs. In particular the Dp-branes are Poincaré dual to the square root of the A-roof genus of the tangent bundle times $ch_{(9-p)/2}$ of the gauge bundle. In the sequel we will ignore the $\hat{A}$ corrections. Thus all D-brane configurations yield a cohomology class, however not every cohomology class is a Chern character of a vector bundle.

For example, consider a D5-brane which is Poincaré dual to some 4-form ω and carries no lower brane charges. Such a D5-brane is allowed if ω is ch_2 of some vector bundle. In the absence of D7-branes, this bundle must have

$$c_1 = 0 \qquad c_2 = \omega\,. \tag{4.1}$$

However for any vector bundle

$$c_3 = c_1 \wedge c_2 + Sq^2 c_2 \bmod 2 \tag{4.2}$$

where "mod 2" means that these two-forms agree as elements of the $\mathbb{Z}_2$ valued cohomology of spacetime. c_1 vanishes and so c_3 must be equal to $Sq^2 c_2$ mod 2. However a Chern class is an element of integral cohomology, and so c_3 must be the lift of $Sq^2 c_2$ to integral cohomology. A differential form has such a lift precisely when it is annihilated by Sq^1 and so the existence of the desired bundle requires

$$0 = Sq^1 Sq^2 \omega = Sq^3 \omega \tag{4.3}$$

where the last equality came from an Adem relation.

Thus we find that a D5-brane configuration corresponds to a section of some bundle on D9-$\overline{\text{D9}}$ pairs exactly if its Poincaré dual is annihilated by Sq^3. Thus Sq^3 is an obstruction to lifting an element of H^4 to K-theory.

In the presence of an H-field we are not interested in K-theory, but rather in twisted K-theory [2] or perhaps the algebraic K-theory of sections of a PU(∞) bundle [16] or $\hat{E}_8$ bundle. In this case the above obstruction is actually

$$d_3 = Sq^3 + H\,. \tag{4.4}$$

This is one of many obstructions, but remarkably only a finite number of obstructions exist for a given differential form and these are all contained in the Atiyah-Hirzebruch spectral sequence (AHSS) reviewed in Subsection 2.1. Physically the failure of this sequence of differentials to annihilate the Poincaré dual of the submanifold wrapped by a D-brane indicates the presence of an anomaly which in the case $p = 3$ is the Freed-Witten anomaly [17].

Not only do some elements of cohomology fail to lift to K-theory, but others are equivalent in K-theory. More precisely, two differential forms are equivalent as K-theory elements if they differ by a form which is in the image of any of the above differentials d_p.

The physical interpretation is that equivalent brane configurations are related by dynamical processes. In IIA only $p = 3$ is nontrivial. In IIB $p = 3$ is nontrivial and also, when $H \neq 0$, $p = 5$ gives a restriction on allowed D5-brane wrappings and identifies states with different numbers of D1-branes.

4.2 The MMS construction

Maldacena, Moore, and Seiberg [7] propose the following classification scheme for physical configurations in the spacetime $M^9 \times \mathbb{R}$ where $\mathbb{R}$ is the time direction. Start with all configurations which are consistent and time-independent, in this case that means that all D-branes wrap submanifolds dual to elements of the kernels of all of the above differential operators. Then identify states that are related by physical processes, that is, identify D-branes wrapping submanifolds whose duals differ by an element of the image of some differential. The final result of this classification scheme is therefore the cohomology with respect to all of the above differential operators, which in IIB is the associated graded algebra of K_0. If instead of branes we considere fluxes then we are supposedly interested in K^1. After solving an extension problem, one arrives at the desired K-group.

This prescription has a simple realization in terms of branes ending on instantonic branes. Here "instantonic" refers either to solutions of the euclidean equation of motion giving rise to tunneling between charge states or time-dependent solutions of the lorentzian equations of motion corresponding to allowed transitions between them. The idea is that an instantonic Dp-brane can sweep out a nontrivial 3-cycle which supports k units of H flux. Assume that the image of the Dp-brane is spin^c or slightly more generally that its dual is annihilated by Sq^3. As we saw above, the classical equations of motion require that k D$(p-2)$-branes end on this Dp-brane. We will see that this is also required for anomaly cancellation on the worldsheet theory of fundamental strings that end on the Dp-brane. The result is that the lone Dp-brane configuration is forbidden and a state consisting of k D$(p-2)$-branes is trivial, as it decays to the vacuum via a process with an instantonic Dp-brane.

In the language of the AHSS, the Poincaré dual (in the 9-dimensional sense) ω of the submanifold wrapped by the Dp-brane is not annihilated by the differential $d_3 = Sq^3 + H$, instead

$$d_3\omega = \eta \tag{4.5}$$

where η is dual (in the 10-dimensional sense) to the submanifold inhabited by the D$(p-2)$-branes. Thus the Dp-brane corresponds to a form that is not d_3-closed and so is forbidden, while the k D$(p-2)$-branes correspond to a form which is d_3-exact and so their configuration is trivial. In particular they can decay into a state with no BPS charges.

It is widely believed that K-theory not only classifies charges, but also classifies fluxes [18]. Technically it is K-cohomology rather than K-homology [19, 5] that is believed to classify fluxes, but this is a subtlety that we ignore throughout this paper as our goal is simply to generalize the spectral sequence, and not to learn what generalization of K-theory our result describes. The allowed RR fluxes are those which satisfy

$$d_3 G_p = Sq^3 G_p + H \wedge G_p = 0\,. \tag{4.6}$$

This is simply the Bianchi identity (3.6) and classical equation of motion (3.7) in the absence of D-brane sources and with a torsion correction. Equation (4.6) is precisely the flux version of the statement discussed above for charges.

4.3 Generalization

As suggested in ref. [14], this result can be generalized to include NS5-branes and fundamental strings. In particular, an NS5-brane can only wrap a 3-cycle with k units of G_3 flux if it is the endpoint of k D3-branes. This is the S-dual of the $p = 5$ case of the previous section. Thus some lone NS5-brane wrappings are not allowed, while some D3-brane configurations that were nontrivial using only the considerations of the previous section are actually trivial because of the dynamical process in which an instantonic NS5-brane appears from the vaccuum, absorbs the D3-branes and then vanishes again.

Including the processes of this section and the last, the total number of D3-branes is conserved only modulo the greatest common divisor of the G_3 and H flux supported on any 3-cycle due to the effects of instantonic NS5-and D5-branes respectively. More generally the D3-branes themselves may have nontrivial wrappings, in which case the relation is more complicated. These effects may be built into a generalization of the AHSS in which one begins with both NS and RR fields and defines d_1 to be the exterior derivative in both cases. Again d_3 is defined on RR fluxes by

$$d_3 G_p = \left(Sq^3 + H\right) G_p \,. \tag{4.7}$$

D3-branes ending on instantonic NS5-branes can be incorporated if one defines the action of d_3 on H as follows:

$$d_3 H = G_3 \wedge H \tag{4.8}$$

where torsion corrections to eq. (4.8) terms will be incorporated in section 5.

There is another relevant process. Recall that a Dp-brane can wrap a p-cycle which supports k units of G_p flux, in which case it needs to be the endpoint of k fundamental strings. Thus some Dp-brane wrappings which seemed consistent according to (4.7) actually prove to be inconsistent. To account for this, one would like to create the differential D such that

$$DG_p \supset G_{8-p} \wedge G_p \tag{4.9}$$

however in this paper we will restrict our attention to the generalizations of d_3 and d_5, which increase the degree of a form by 3 and 5 respectively.

The degree 3 case of (4.9) occurs only for the case $p = 5$, corresponding to D3-branes wrapping 3-cycles that support k units of G_3 flux. These branes must be the endpoints of k fundamental strings, which is the S-dual of the statement from the previous section with $p = 3$. Thus the embedding of a D3-brane has two consistency conditions, one coming from the H flux of a 3-cycle that it wraps and one from the G_3 flux. p units of H flux and q of G_3 flux require that a (p, q) string end on the D3-brane. This is summarized by requiring the vanishing of the differential d_3 on the G_5 flux surrounding a D3-brane:

$$d_3 G_5 = G_3 \wedge G_5 \tag{4.10}$$

where again torsion corrections will be incorporated in section 5.

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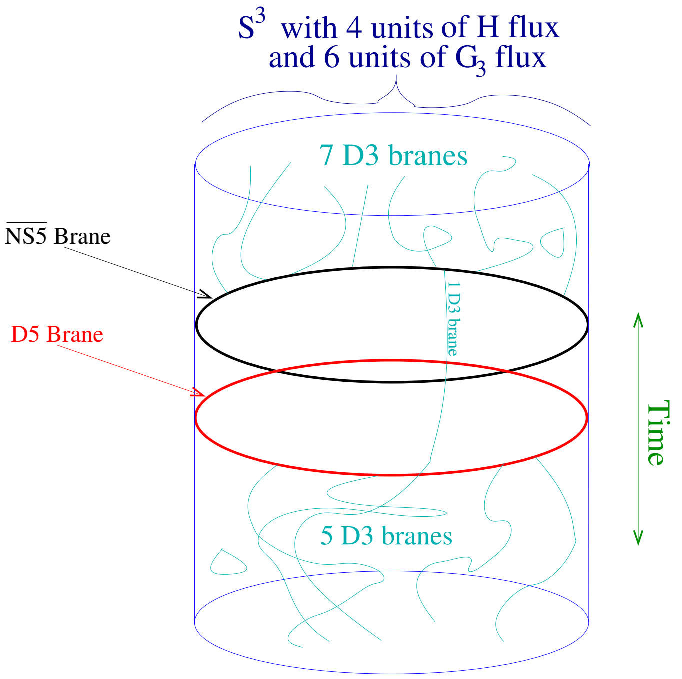

Figure 2: A D5-brane and $\overline{\text{NS5}}$ wrap a three sphere that supports 4 units of H flux and 6 of G_3 flux. Anomaly cancellation requires that 4 D3-branes end on the D5-brane and 6 begin on the $\overline{\text{NS5}}$. Therefore one allowed process begins with 5 D3-branes, 4 of which decay via the instantonic D5-brane leaving just 1. Later the instantonic $\overline{\text{NS5}}$ appears and disappears, leaving 6 more D3-branes for a total of 7. Thus we see that the number of D3-branes is only conserved modulo 2, where 2 is the greatest common divisor of 4 and 6.

5. Torsion corrections

5.1 The Freed-Witten anomaly

Consider a gauge theory on a manifold M with gauge group G. A fermion that transforms in the fundamental representation of the gauge group is a section of the bundle

$$S(M) \otimes G \tag{5.1}$$

where $S(M)$ is a spin "bundle" and G is the associated vector "bundle" on which some gauge field A is a connection. The word "bundle" appears in quotes because charged fermions may exist even in the absence of a spin bundle, that is on a manifold that is not spin. More precisely, the "bundle"s in (5.1) may have transition functions whose triple products are not the identity, so long as the triple products in $S(M)$ and G cancel so that the tensor product (5.1) is an actual bundle. Such a bundle defines a spin^c structure on M, and so we see that M must be spin^c if the gauge theory contains charged fermions.

The gauge theory on the worldvolume of a D-brane does contain charged fermions, arising from the endpoints of fundamental strings. Therefore D-branes are restricted to wrap spin^c submanifolds of spacetime which are submanifolds $N \subset M$ whose tangent bundle's third Stieffel-Whitney class vanishes,

$$W_3(N) = 0\,. \tag{5.2}$$

The Steenrod square Sq^3 is a map from p-forms ω to $(p{+}3)$-forms η that takes the Poincaré dual of N to itself wedged with W_3 of the normal bundle of N pushed forward onto M. This map is not necessarily an injection and so

$$i_*(W_3(N)) \equiv Sq^3(PD(N)) = 0 \tag{5.3}$$

is a necessary but not sufficient condition for the satisfaction of eq. (5.2). We will see in an example below that combining (5.3) with a degree five differential operator better approximates the condition (5.2).

We have just seen that the differential forms Poincaré dual to a configuration of D-branes must be annihilated by Sq^3. Similarly [18] the corresponding fieldstrengths must be annihilated by Sq^3.

5.2 Freed-Witten with H flux

In the presence of a nontrivial H-field the situation becomes slightly more complicated. Although the answer appeared in [2], the answer was later justified in [17] by an analysis of a global anomaly in the worldsheet path integral of an open string ending on a D-brane.

In the absence of a B-field the path integral measure for a string with worldsheet Σ contains the terms

$$\mathrm{pfaff}(D)\exp\left(i\oint_{\partial\Sigma} A\right) \tag{5.4}$$

where D is the worldsheet Dirac operator and A is the gauge potential of the worldvolume U(1) gauge theory on the D-brane. The authors showed that it suffices to consider the case in which the same spin structure is used for left movers and right movers. In this case the Dirac operator is real and so pfaff(D) must be real.

The pfaffian is real, but there is no natural way to determine its sign. Instead one may try to choose a sign, but there may be no consistent sign choice. That is, it may be that whatever sign one chooses, if the worldsheet slides along a particular circle and returns to its original position the sign may flip. In fact the authors proved that the sign flips

$$\alpha = \int_S w_2(N) \tag{5.5}$$

times, where S is the surface traced out by $\partial\Sigma$ and $w_2(N)$ is the second Stieffel-Whitney class of the submanifold into which the brane is embedded. Integration is used to denote the natural homology-cohomology pairing.

Thus the path integral measure is only well defined if the holonomy $\exp(i\oint A)$ changes signs α (mod 2) times as well. In this case A could not actually be the connection on a bundle, as this would imply that $\alpha = 0$, but rather A is the connection on a "bundle". The fact that both terms in (5.4) yield cancelling contributions implies that the tensor product of the spin "bundle" and the "bundle" on which A is a connection gives an actual bundle.

Just as in the last subsection, this bundle provides a spin^c structure. The worldvolume and worldsheet approaches in these two subsections had to agree, as charged fermions on the D-brane worldvolume are the result of such worldsheets. In particular, such an A exists precisely when N is spin^c or equivalently $W_3(N) = 0$.

In the presence of a B-field (5.4) becomes

$$\mathrm{pfaff}(D) \exp\left(i \oint_\Sigma B\right) \exp\left(i \oint_{\partial\Sigma} A\right). \tag{5.6}$$

Now anomaly cancellation requires that A be chosen so that the change in holonomy $\exp(i \oint A)$ precisely cancels the change in the product of the first two terms. The obstruction to the existence of such an A is no longer simply $W_3(N)$, but there is a new correction arising from $H = dB$. The holonomy now needs to change signs $\int(w_2 + B)$ times and so for a general H the condition for the existence of A is now

$$\beta(w_2 + B) = W_3 + H = 0\,. \tag{5.7}$$

Thus a D-brane wraps a spin^c submanifold N if and only if N carries trivial H, however a D-brane can instead wrap a submanifold N which is not spin^c in the presence of a nonvanishing H which is precisely equal to $W_3(N)$. In particular $2H = 0$ on any wrapped submanifold.

5.3 Strings and NS5-branes

We have seen that D-branes can only wrap submanifolds that are spin^c, or more generally, that satisfy (5.7). This analysis is readily extended to fundamental strings and NS5-branes. First, the restriction is trivial in the case of fundamental strings as they sweep out two dimensional surfaces, which are automatically spin^c. There is no analog of the H term because there are no objects that end on fundamental strings in the weak coupling limit of type-II.

While we do not know how to extend this argument to the NS5-branes of IIA, in IIB NS5-branes host worldvolume 5+1 dimensional U(1) gauge theories. These theories have charged fermions arising from the ends of D-strings. Thus we expect NS5-branes to wrap spin^c submanifolds. This was seen in an example in ref. [20] and in more generality in ref. [21]. However the worldvolume of the D-strings couples to the 2-form C_2 and so if this C_2 is nontrivial then one may expect that worldsheet anomaly cancellation on the D-string provides a correction to (5.4). In particular if one trusts the S-dual of the argument above one may suspect that

$$W_3 + G_3 = 0 \tag{5.8}$$

on the worldvolume of the NS5-brane.

Applying this relation to the H flux whose source[2] was the NS5-brane, one would find

$$\left(Sq^3 + G_3\right) H = 0\,. \tag{5.9}$$

[2]The arguments of ref. [18] suggest that such relations on sources also apply to the fields that they create.

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As evidence for (5.9) notice that in the supergravity approximation it reduces to $G_3 \wedge H = 0$, which at the level of cohomology is a supergravity equation of motion (3.6). Eq. (5.9) will serve only as a motivation for our conjecture.

5.4 Condition on G_3 and H

To motivate the condition eq. (2.11) on the pair G_3 and H, let us review the relevant pieces of our argument. First we know that in the supergravity limit torsion corrections can be neglected and so the supergravity equation of motion

$$G_3 \wedge H = 0 \quad \text{when} \quad G \wedge G = H \wedge H = 0 \tag{5.10}$$

holds at the level of rational cohomology. Also we use the fact that there are fermions charged in a U(1) gauge theory on the worldvolume a single D5 or NS5-brane with no background fluxes to arrive at

$$W_3 = 0 \quad \text{when} \quad H = 0 \tag{5.11}$$

on a D5-brane worldvolume (the Freed-Witten anomaly) and

$$W_3 = 0 \quad \text{when} \quad G = 0 \tag{5.12}$$

on an NS5-brane worldvolume. These imply that Sq^3 annihilates the Poincaré duals of the worldvolumes. Following the reasoning of [18] this translates into restrictions of the corresponding fieldstrengths:

$$Sq^3(G_3) = 0 \quad \text{when} \quad H = 0 \qquad Sq^3(H) = 0 \quad \text{when} \quad G_3 = 0 \tag{5.13}$$

respectively.

Furthermore we know [17] that a D5-brane can be wrapped on a manifold with $W_3 \neq 0$ when there is a background H that precisely cancels this W_3. This suggests that the generalization of the above formulas involves mod 2 additions of Steenrod squares and fluxes so that they may cancel each other. There is not enough information to specify the generalization completely, thus the final form will be a conjecture. Neglecting G_3 and H independent terms,[3] there are two natural guesses for the desired condition.
Condition A:

$$\left(Sq^3 + H\right) G_3 = \left(Sq^3 + G_3\right) H = 0\,. \tag{5.14}$$

Condition B:

$$G \wedge H_3 + Sq^3(G_3 + H) = 0\,. \tag{5.15}$$

Notice that only condition B is S-duality invariant. Condition B has also appeared in [13] as a possible generalization. In fact, the authors verified it via the M-theory partition function in the case of IIB configuarations that can be obtained via the compactification of M-theory on a torus. Below we will provide an example which appears to exclude[4] condition A but is consistent with B.

[3] We have no reason to disallow such terms, except that they have not been seen in examples.

[4] More precisely it excludes a worldvolume version of A in terms of W_3's.

6. Evidence and SU(3)

6.1 NS5-brane backreaction

String perturbation theory breaks down in the near-horizon region of an NS5-brane when g_s is small at infinity. This means that one requires extra care when formulating arguments concerning the physics of this region. Here we present three reasons to trust the arguments presented above.

First, as already noted, in every case but one the restriction is simply a well-known supergravity equation of motion with the known torsion correction from the Freed-Witten anomaly. Our only contribution is to interpret these equations in the MMS framework so as to show the compatibility of the K-theory framework with S-duality at the level of d_3.

Second, the supergravity arguments are formulated on a sphere consisting of points at any fixed distance from the NS5-brane, in particular a distance can be chosen to be large enough so that string perturbation theory is valid. The fact that the signed intersection number of D3-branes with this tube is equal to the G_3 flux on a 3-cycle wrapped by the NS5-brane indicates that the 3-branes must thread down the throat of the NS5-brane as desired.

Finally, the torsion arguments are the result of the anomaly structure of the NS5-brane worldvolume theory. In particular, U(1) charged fermions require a spin^c structure, or equivalently the pfaffian of the Dirac operator must be well defined. Such arguments tend to be stable under deformations of the theory. In particular, one could increase g_s thereby smoothing the geometry around the NS5-brane and then make the same arguments, which one expects to still hold when the asymptotic string coupling is turned back down. One may still be concerned that strong coupling effects near the NS5-brane lead to additional terms in the D-string worldvolume that may cancel this anomaly, such as the G_3 term which can cancel the anomaly in the case of an NS5-brane in a RR background. If such terms do exist then perhaps one may learn about them by understanding when this anomaly argument fails.

In addition to the near-horizon geometry of NS5-branes, there is also reason to be concerned about the validity of S-duality, particularly in cases with less than 16 supercharges and torsion cohomology classes [6]. For this reason we do not invoke S-duality in such cases. Yet our conjecture does turn out to be consistent with S-duality.

6.2 (1,1) 5-branes on non-Spin^c manifolds

Consider a $(1,1)$ 5-brane wrapped around a submanifold $N \subset M$ with no background fluxes that restrict nontrivially to the submanifold. The brane will create fluxes $G_3 = H$ and so in particular the conditions on the fluxes (5.14) and (5.15) can be reexpressed Condition A:

$$0 = Sq^3 G_3 + H \wedge G_3 = G_3 \wedge G_3 + G_3 \wedge G_3 = 2G_3 \wedge G_3 \tag{6.1}$$

$$0 = Sq^3 H + H \wedge G_3 = H \wedge H + H \wedge H = 2H \wedge H \tag{6.2}$$

Condition B:

$$0 = Sq^3(G_3 + H) + H \wedge G_3 = G_3 \wedge G_3 + H \wedge H + G_3 \wedge H = 3G_3 \wedge G_3 = Sq^3(G_3)\,. \tag{6.3}$$

In particular condition A is always satisfied. Condition B is not satisfied precisely when $Sq^3G_3 \neq 0$. In particular if condition B is not satisfied then N is not spin^c, because Sq^3G_3 is a pushforward of $W_3(N)$.

However a (1,1) 5-brane carries a U(1) gauge field and has charged fermions corresponding to (1,1) strings that end on it.

Thus the worldvolume must have a spin^c structure so that the Dirac operator for these fermions can be constructed. This means that in fact there is an anomaly if a (1,1) 5-brane wraps a non-spin^c submanifold. And so we learn that whenever $W_3(N)$ is not in the kernel of the pushforward i_* of the inclusion $i : M \hookrightarrow N$, the anomaly is predicted by condition B but not predicted by condition A, which is always satisfied for a (1,1) 5-brane with no background fluxes. This is yet another piece of evidence in favor of condition B, which is the condition chosen in our conjecture.

While unfortunately i_* does kill $W_3(N)$ for all submanifolds N that the authors know how to wrap branes around,[5] we will now illustrate an example of this (1,1) 5-brane anomaly in a case where $W_3(N) \neq 0$ but $Sq^3 = 0$.

6.3 The topology of SU(3)

We will consider type-II string theory on $\mathrm{SU}(3) \times \mathbb{R}^{1,1}$. We will begin with type-IIA, reviewing the results of [7] and then will employ T-duality to understand the scenario of interest. First we will describe the relevant features of the topology of the SU(3) group manifold.

The nonvanishing integral homology classes of SU(3) are

$$H_0(\mathrm{SU}(3), \mathbb{Z}) = \mathbb{Z}\,, \quad H_3(\mathrm{SU}(3), \mathbb{Z}) = \mathbb{Z}\,, \quad H_5(\mathrm{SU}(3), \mathbb{Z}) = \mathbb{Z}\,, \quad H_8(\mathrm{SU}(3), \mathbb{Z}) = \mathbb{Z}\,, \tag{6.4}$$

which are represented by the submanifolds p, S^3, M_5, and SU(3) respectively, where p is a point, S^3 is an embedded SU(2) and M_5 is the group manifold $\mathrm{SU}(3)/\mathrm{SO}(3)$. Similarly the integral cohomology ring is trivial except for H^0, H^3, H^5, and H^8 which are isomorphic to $\mathbb{Z}$ and generated by 1, x_3, x_5, and x_8 which are Poincaré dual to SU(3), M_5, S^3, and p respectively. The $\mathbb{Z}_2$ homology and cohomology classes are the same with $\mathbb{Z}$ replaced by $\mathbb{Z}_2$.

The submanifold M_5 is more interesting. This has nonvanishing integral homology classes

$$H_0(M_5, \mathbb{Z}) = \mathbb{Z} \quad H_2(M_5, \mathbb{Z}) = \mathbb{Z}_2 \quad H_5(M_5, \mathbb{Z}) = \mathbb{Z} \tag{6.5}$$

with generators q, M_2 and M_5 respectively where $M_2 \subset M_5$ is a 2-sphere. By the universal coefficient theorem the $\mathbb{Z}_2$ homolgy is similar

$$H_0(M_5, \mathbb{Z}_2) = \mathbb{Z}_2 \quad H_2(M_5, \mathbb{Z}_2) = \mathbb{Z}_2 \quad H_3(M_5, \mathbb{Z}_2) = \mathbb{Z}_2 \quad H_5(M_5, \mathbb{Z}_2) = \mathbb{Z}_2 \tag{6.6}$$

where $H_3(M_5, \mathbb{Z}_2)$ is generated by an $M_3 \subset M_5$ whose boundary wraps the 2-sphere M_2 twice. The integral cohomology classes are

$$H^0(M_5, \mathbb{Z}) = \mathbb{Z} \quad H^3(M_5, \mathbb{Z}) = \mathbb{Z}_2 \quad H^5(M_5, \mathbb{Z}) = \mathbb{Z} \tag{6.7}$$

[5] An exception may be the $\mathbb{RP}^5$ in ref. [20], but in our paper, which is about IIB, we do not consider configurations with orientifolds.

where the generator of $H^3(M_5,\mathbb{Z})$ is $W_3(M_5)\neq 0$, the third Stieffel-Whitney class of M_5. In particular M_5 is not spin^c. The $\mathbb{Z}_2$ cohomology is

$$H^0(M_5,\mathbb{Z}_2)=\mathbb{Z}_2 \quad H^2(M_5,\mathbb{Z}_2)=\mathbb{Z}_2 \quad H^3(M_5,\mathbb{Z}_2)=\mathbb{Z}_2 \quad H^5(M_5,\mathbb{Z}_2)=\mathbb{Z}_2 \tag{6.8}$$

where $H^2(M_5,\mathbb{Z}_2)$ is generated by $w_2(M_5)$, the second Stieffel-Whitney class.

The authors consider a background H-field of $H=kx_3$. They then show that this H-field restricts nontrivially to M_5:

$$x_3|_{M_5}=W_3(M_5)\,. \tag{6.9}$$

They conclude that if a D6-brane wraps M_5 precisely once, the condition for worldvolume anomaly cancellation is the constraint

$$0=W_3(M_5)+H|_{M_5}=(1+k)W_3(M_5) \tag{6.10}$$

which is satisfied precisely when k is odd.

6.4 A (1,1) 5-brane on M_5

Compactify the noncompact spatial direction on a circle, T-dualize with respect to it and then take the radius back to infinity. This gives the same configuration as above but in IIB. Now one can wrap a (1,1) 5-brane around M_5. Notice that (1,1) strings may end on the 5-brane and they will yield fermions charged under the U(1) worldvolume gauge group. In the absence of any external fluxes that might affect the path integral of the (1,1) string, anomaly cancellation requires that the submanifold wrapped by the 5-brane be spin^c. However M_5 is not spin^c and so this configuration is anomalous.

So we see that $W_3\neq 0$ and there is an example of the kind of anomaly employed throughout this paper. Unfortunately W_3 is annihilated by i_* and so $Sq^3=0$, thus the anomalous configuration is not excluded by d_3. However, as seen in the T-dual situation in ref. [7], the anomaly is detected by d_5.

6.5 The MMS instanton and d_5

Consider IIB on $\mathrm{SU}(3)\times\mathbb{R}^{1,1}$ with a background flux $H=kx_3$. Following ref. [7], anomaly cancelation on the worldvolume of an instantonic D7-brane wrapped around the SU(3) implies that such instantons are the endpoint of k D5-branes which each wrap M_5. Thus a state consisting of a multiple of k such D5-branes can dynamically decay to the vacuum. Similarly one can wrap an instantonic D3-brane around $M_3\times\mathbb{R}$ and anomaly cancellation requires k D1-branes extended along $\mathbb{R}$ to end on it. Thus D1-brane number is also, at most, conserved modulo k.

To summarize, after taking the cohomology with respect to $d_3=Sq^3+H$, we find that D5-branes wrapped on M_5 are classified by $\mathbb{Z}_k$ and D1-branes wrapped on $\mathbb{R}$ are also classified by $\mathbb{Z}_k$. However M_5 is not spin^c, therefore if a D5-brane wraps M_5 r times, the anomaly cancellation condition on the brane worldvolume is

$$0=r(W_3+H)=r(1+k)\,. \tag{6.11}$$

When r is even this is always satisfied and so an even number of wrappings is always permitted. However an odd number of wrappings is anomalous whenever k is even. Thus these D5-branes are actually only classified by elements of $2\mathbb{Z}_k = \mathbb{Z}_{k/2}$, in disagreement with the above calculation obtained using only d_3. Notice that when H and so k vanishes, the D5-branes continue to be classified by $2\mathbb{Z} = \mathbb{Z}$, although half of the wrappings that one would expect become anomalous. Thus one would like to find that d_5 is nontrivial precisely when k is even and $k > 0$.

As a hint, we examine instantonic D5-branes wrapped on $M_5 \times \mathbb{R}$. The anomaly cancellation condition is

$$0 = W_3(M_5) + H|_{M_5} = (1 + k)x_3 \tag{6.12}$$

where x_3 is the generator of $H^3(M_5, \mathbb{Z}_2) = \mathbb{Z}_2$. This is satisfied when k is odd. When k is even, anomaly cancellation requires an instantonic D3-brane whose boundary is $M_2 \times \mathbb{R} \subset M_5 \times \mathbb{R}$. Recall that M_2 is the generator of $H_2(M_5, \mathbb{Z}) = \mathbb{Z}_2$ and so it is not a boundary in M_5. However $H_2(\mathrm{SU}(3)) = 0$ and so M_2 must bound a 3-manifold X in SU(3). The instantonic D3-brane wraps some $X \times \mathbb{R}$. We have seen that the boundary of M_3 is two copies of M_2 and in fact two copies of X can be deformed to M_3. Moreover the H flux restricted to X is precisely half of the total k units of H flux.[6]As X is a 3 manifold, it is trivially spin^c and so anomaly cancellation only demands that $k/2$ D1-branes extended along $\mathbb{R}$ must end on the D3-brane. Therefore this double-instanton violates D1 charge by $k/2$ units, meaning that like D5-branes wrapped on M_5, D1-branes are classified by $\mathbb{Z}_{k/2}$. Again this disagrees with the result obtained by simply taking the cohomology with respect to d_3. As recognized in ref. [7], the reason for this discrepency is that d_5 acts via

$$d_5(x_3) = \frac{k}{2} x_5 \cup x_3 = \frac{k}{2} x_8\,. \tag{6.14}$$

It would be desirable to have a formula in terms of a cohomological operation, so that it may generalize to other examples.

The crucial observation is that this contribution to d_5 is the result of an instanton with a secondary instantonic brane that is bounded by M_2 which is Poincaré dual in M_5 to $W_3(M_5) + H$. The fact that the boundary is PD to $W_3 + H$ is independent of the example, it was the condition for anomaly cancellation on the primary instantonic brane. The secondary instanton wraps "half of M_3", which is half of the PD in M_5 to $w_2 + B$. Again this fact is example independent as a result of the following argument. The primary instanton's anomaly required that $H^3(M_5, \mathbb{Z})$ contain a 2-torsion class $W_3 + H$. The fact that this is a 2-torsion class implies that there is some 2-cochain b whose coboundary is precisely twice the 3-cochain c corresponding to $W_3 + H$. Now consider the dual chain complex and let β be the dual 2-chain to b and γ the dual 3-chain to c. Then $\partial(c) = 2b$.

[6]More precisely there must be $k/2$ units of dF on the D3-brane worldvolume or else the 5-brane will still be anomalous because

$$0 \neq \int_{M_5} dF \wedge F = \int_{\partial M_5 = 0} F \wedge F = 0\,. \tag{6.13}$$

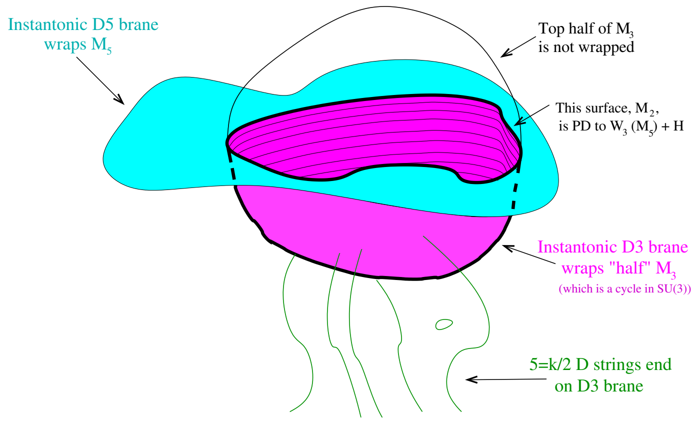

Figure 3: The double-instanton of Maldacena, Moore, and Seiberg for $k = 10$. The primary instanton is an instantonic D5-brane which wraps $M_5 \times \mathbb{R}$ while a secondary instantonic D3-brane wraps $\mathbb{R}$ crossed with a 3-cell in SU(3). This 3-cell is bounded by M_2, the Poincaré dual of $W_3(M_5) + H$ in M_5. Anomaly cancellation on the secondary instanton implies that it must devour $k/2 = 5$ D-strings extended along $\mathbb{R}$.

The boundary of the secondary instanton wraps b and so the rest of the instanton must wrap $c/2$, an expression which only makes sense when interpreted in terms of the total spacetime which, being spin^c itself, may be required to have such a cycle. If the total spacetime does not have such a cycle, then this double-instanton cannot exist and thus the contribution to d_5 is zero.

To review, the double-instanton consists of a primary instanton which wraps a submanifold N and a secondary instanton whose boundary wraps a submanifold of N PD to $W_3(N) + H$. The worldvolume of the secondary instanton is $c/2$ where c is the homology class PD to $w_2(N) + B$ and $c/2$ is interpreted as a chain in the total space. The secondary instanton itself is subject to anomaly cancellation. That is

$$W_3\left(\frac{c}{2}\right) + H|_{c/2} = 0\,. \tag{6.15}$$

$H|_{c/2}$ is defined to be one half of $H|_c$, whose 2-torsion is determined by the boundary conditions that must be imposed upon the secondary instanton's gauge bundle to kill the primary instanton's anomaly. We do not know what boundary conditions these are, and so we have not defined the 2-torsion part of eq. (6.15).

In our example and the T-dual example of ref. [7], the dimension of the secondary instanton is small enough that the W_3 term can be ignored and we will ignore it for now. It seems quite possible that this happens in general. This leaves

$$0 = H|_{c/2} = \frac{1}{2}H|_c = \frac{1}{2}(H \cup (w_2(N) + B))|_N\,. \tag{6.16}$$

We would like a cohomological formula in terms of the PD of N in the total space, or in terms of the dual fieldstrength of a D-brane wrapped on N. To obtain this, we simply pushforward eq. (6.16) onto the total space $i : N \hookrightarrow M$. This yields the condition

$$\frac{1}{2}i_*(H \cup (w_2(N) + B)) = 0\,. \tag{6.17}$$

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Thus the contribution of the double-instanton to d_5 appears to be

$$d_5(\omega) = \frac{1}{2} i_*(H \cup (w_2(\mathrm{PD}(\omega)) + B))\,. \tag{6.18}$$

If division by two is not possible, this double-instanton must create a state with half a unit of D-string charge which would violate the Dirac quantization condition. Therefore in such a case the anomaly appears to be incurable in a compact space, although if a tendril extends to spatial infinity perhaps a compensating fractional charge may lie at the end. Fortunately the arguments above appear to suggest that division by 2 is always possible (in fact it's genrically too possible and so ill-defined).

Notice that in the $\mathrm{SU}(3) \times \mathbb{R}^{1,1}$ case

$$d_5 = \frac{1}{2} H \cup Sq^2\,. \tag{6.19}$$

Like (6.18), this formula is difficult to decipher. Applying d_5 to x_3 we find the cup product of H with $Sq^2(x_3)$. This cup product is well defined in $\mathbb{Z}_2$ cohomology, but we do not want to restrict H to $\mathbb{Z}_2$ cohomology because we need $k/2$. Instead we interpret this expression as follows. Include $\mathbb{Z}_2 = H^5(\mathrm{SU}(3), \mathbb{Z}_2) \hookrightarrow H^5(\mathrm{SU}(3), \mathbb{Z}) = \mathbb{Z}$ such that the inclusion is the identity modulo 2. This is not a homomorphism and it is not canonical. Different choices of inclusion will yield different expressions for d_5, but these expressions will differ by H cupped with a 2-cocycle. However recall that we have quotiented by $d_3 = H$ and so in this quotient ring the different possible expressions for d_5 are all equal and so (modulo the choice of division by 2 discussed above) d_5 is well defined, although it is not well defined on $H^*(\mathrm{SU}(3))$.

We do not claim that this is a complete expression for d_5, merely that it is the contribution to d_5 from the double-instanton of ref. [7]. In particular, d_5 can also contain mod 3 cohomology operations like βP^2, which may be calculable from $\mathbb{Z}_3$ phases in M-theory such as those in refs. [12, 15].

6.6 d_5 with S-duality and an example

We expect that the reinclusion of NS5-branes and background RR flux will allow an S-duality invariant generalization of the above formula for d_5. In particular there should be two conditions in such a generalization, reflecting the requirements that both the number of fundamental strings and the number of D-strings ending on the double instanton vanish. Rather than presenting a general formula, we will simply investigate the example of IIB string theory on $\mathrm{SU}(3) \times \mathbb{R}^{1,1}$. Again we will include a background H flux $H = kx_3$ but now we will also include a background G flux $G_3 = lx_3$. Let both k and l be nonzero.

As in ref. [7], Sq^3 acts trivially on the cohomology ring of SU(3) and so d_3 acts simply by multiplication by G or H. This means that any brane which wraps the S^3 in SU(3) is anomalous. In particular, an instantonic D3-brane wrapping this $S^3 \times \mathbb{R}$ will be the endpoint of k D-strings and l F-strings. Thus, neglecting the effects of d_5, D-strings are classified by $\mathbb{Z}_k$ and F-strings by $\mathbb{Z}_l$.

A similar classification for 5-branes is more difficult because of our ignorance about the nature of D7-branes. Fortunately, such a classification will not be necessary for the rest of the example. An instantonic D7-brane that wraps SU(3) will be the endpoint of k D5

k	l	D5-Instanton	NS5 Instanton	(p,q) strings
Odd	Odd	No	No	$\mathbb{Z}_k \times \mathbb{Z}_l$
Odd	Even	No	Single	$\mathbb{Z}_k \times \mathbb{Z}_l$
Even	Odd	Single	No	$\mathbb{Z}_k \times \mathbb{Z}_l$
Even	Even	Double	Double	$(\mathbb{Z}_k \times \mathbb{Z}_l)/\mathbb{Z}_2$

Table 1: Possible Instantons and String Classification.

branes and so D5-branes are classified, at the level of d_3, by $\mathbb{Z}_k$. If one trusts S-duality in such a setting (and only pretended to care about weak coupling) then there will be a similar instanton involving a 7-brane and l NS5-branes, and so NS5-branes will be classified by $\mathbb{Z}_l$, ignoring again the fact that some of these configurations may not turn out to be d_5 closed.

Thus far we have obtained an approximate classification using only the differential d_3. A more thorough classification is obtained by including the double-instanton. For concreteness, we will restrict our attention to $(1,0)$, $(1,1)$, and $(0,1)$ 5-branes. Notice that a $(1,1)$ 5-brane is only relevant if it gives rise to a single instantonic D3-brane, but in this case an instantonic D5 and an instantonic NS5 would have yielded the same D3-brane. Thus the $(1,1)$ 5-brane does not need to be considered separately to classify (p,q) strings in this example. This leaves us with only two double-instantons to consider, which have primary instantons that are a single D5-brane and a single NS5-brane respectively.

Imagine that k and l are both odd. In this case the induced H and G flux on $H^3(M_5,\mathbb{Z}) = \mathbb{Z}_2$ are each equal to $W_3(M_5)$. In particular this means that $W_3 + H = W_3 + G = 0$ on M_5 and so a single D5-brane or NS5-brane instanton is not anomalous and there is no double instanton. Thus the 5-branes and strings continue to each be classified by $\mathbb{Z}_k \times \mathbb{Z}_l$.

Next try k even and l odd. Thus $W_3 + G = 0$ on the worldvolume of an NS5-brane and so if one trusts S-duality[7] then again there is no double-instanton with an NS5-brane. Of course, since $W_3 + H \neq 0$ odd numbers of D5-branes are forbidden, leaving D5-branes to be classified by $\mathbb{Z}_{k/2}$. In addition, one seems to get a double-instanton consisting of a D5 and a D3-brane. Anomaly cancellation on this D3-brane would appear to require $k/2$ D-strings and $l/2$ F-strings. However $l/2$ is not an integer and so this double instanton must be absent. Thus, unlike the case of l even, D-strings are classified by $\mathbb{Z}_k$ rather than $\mathbb{Z}_{k/2}$.

This is alarming as we know of no physical mechanism which prevents a D5 from sweeping out M_5, but in this process it would create half-integral F-string charge. One possible resolution might be that the partition function for such a process vanishes, as it does in general when there is a Freed-Witten anomaly [25]. Another possibility is that this background is forbidden. If k is odd and l is even the story is the S-dual of the above case.

The last possibility is that k and l are both even. Now there are double-instantons consisting of both an NS5-D3 pair and a D5-D3 pair. Both instantons violate (p,q) string charge by $(l/2, k/2)$. Thus (p,q) strings are classified by $(\mathbb{Z}_k \times \mathbb{Z}_l)/\mathbb{Z}_2$. Notice that the $(1,1)$ fivebrane yields the same double-instanton. This is a manifestation of the nonlinearity of the anomaly cancellation condition. In general this perspective does not tell us which (p,q) five branes are anomalous and so does not allow us to classify (p,q) 5-branes.

The above results are summarized in table 1

[7]Our goal in this subsection is not to test S-duality, but rather to learn what it can tell us about d_5.

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7. M theory

A systematic analysis of the above example may lead to an S-duality covariant extension of d_5, but what variant of K-theory does this describe? Perhaps the best source of clues as to the nature of any such mysterious variant of K-theory is M-theory. The separation of fields into NS and RR is a result of the way in which M-theory is compactified and so an understanding of the classification of these fields is likely to be a by-product of a classification of fields in M-theory combined with the action of Kaluza-Klein reduction on this classification.

In M-theory there is no sense in which we can work at weak coupling. Nonetheless we will use the MMS prescription to classify M2 and M5-brane configurations. In the absence of any understanding of the quantum theory on these branes we will refrain from discussing the torsion part of this problem, and instead concern ourselves with the supergravity approximation. Distances considered will be much larger than the 11-dimensional Planck scale.

The 11-dimensional supergravity action contains the terms

$$S \supset \int G_4 \wedge \star G_4 + C_3 \wedge G_4 \wedge G_4 \qquad G_4 = dC_3\,. \tag{7.1}$$

This leads to the equation of motion

$$d * G_4 = G_4 \wedge G_4 \tag{7.2}$$

which, similarly to the D-brane case, implies that an M5-brane wrapped around a 4-cycle that supports k units of G_4 flux is the endpoint of k M2-branes.

This suggests that a spectral sequence begins with

$$E_1 = E_1^5 \oplus E_1^8 = H^5(M,\mathbb{Z}) \oplus H^8(M,\mathbb{Z}) \tag{7.3}$$

and has a single differential

$$d_4 = G_4 \tag{7.4}$$

which is clearly trivial on H^8 but need not be trivial on H^5:

$$d_4 : H^5(M) \longrightarrow H^8(M) \otimes_Z H^1(\mathbb{R}) \cong H^8(M)\,. \tag{7.5}$$

What is this a spectral sequence for? Following [12, 13] one can interpret G_4 as p_1 of an E_8 bundle over M. The above authors considered a bundle not on the 11-dimensional manifold, but on a 12-dimensional auxillary manifold that the 11-dimensional manifold bounds. However the bundle can be restricted to the 11-dimensional manifold and in fact the authors proved that the choice of 12 manifold is irrelevant.

$\pi_3(E_8) = \mathbb{Z}$ and all other $\pi_{n<15}(E_8) = 0$. Thus E_8 bundles on manifolds with dimensions of less than 16 are classified by their first Pontrjagin class p_1, reflecting the nontrivial E_8 bundles over S^4's in the 4-skeleton of M where the transition function on the S^3 equator is an element of $\pi_3(E_8)$. This p_1 can be identified with G_4 resulting in the following interpretation of M5-branes.[8]

M5-branes are the defects in the E_8 bundle such that the restriction of the bundle to an S^4 linking an M5-brane once is the elementary E_8 bundle described in the previous paragraph. As the other homotopy classes of E_8 vanish, M5-branes will be the only such topological defects. M2-branes arise as the electromagnetic dual of the M5-branes. In particular the existence of M2-branes is necessitated by the above anomaly for M5-branes wrapped around cycles of nonvanishing G_4 flux. The existence of M2-branes is also required by the Hanany-Witten transition, which requires an M2-brane to be created when two M5-branes cross, as follows from the above supergravity equation of motion.

Alternately the supergravity equation of motion

$$d * G = G \wedge G \tag{7.6}$$

indicates that M2-branes are dual to p_2 of the gauge bundle, although for an E_8 bundle the relation

$$p_2 = p_1 \wedge p_1 \tag{7.7}$$

reveals that there is no new topology in this characteristic class.

Therefore M2 and M5-branes can be classified by E_8 bundles, and somehow eqs. (7.3) and (7.4) are the beginning of an analog of a spectral sequence for a classification of such bundles. However we do not know if one should look at a classification of a single E_8 bundle, or a construction more like K-theory where one looks at equivalence classes of pairs of E_8 bundles, reflecting annhilation of pairs of M10-branes. M10-branes were conjectured to exist in [24] and as evidenced in [4] they may carry 11-dimensional vector multiplets and become the unstable D9 sphalerons of IIA after compactification on a circle. The above classification scheme may suggest that each M10-brane must carry 248 vectormultiplets.

8. Conclusion

We have constructed rules that allow us to calculate E_3 for a given manifold and even E_N if one can find the higher differentials by analyzing dynamical processes. A promising testing ground for our proposal is IIB on the SU(3) group manifold. Above we have begun to investigate this example, but open problems remain. In particular some processes appear to lead to violations of the Dirac quantization condition. In addition it is not known what torsion characteristic classes of a secondary instanton will cancel the Freed-Witten anomaly on a primary instanton. Perhaps insight into this question may be gained by extending away from the weak-coupling limit, so that both instantons and smoothly connected. Then the gauge bundle must smoothly extend as well, and the gauge bundle over the combined

[8]Recall that G_4 is itself equal to w_4 of the tangent bundle modulo 2. This is remenicent of the 10-dimensional condition in heterotic M-theory [22, 23]. If one interprets the gauge bundle over the end of the world as a restriction of the E_8 gauge bundle on the 11-dimensional space to its boundary boundary, then it is possible that the 11-dimensional condition, which arises from membrane worldvolume anomaly cancellation, restricts to the 10-dimensional condition, which arises from a 10-dimensional gravitational anomaly.

object tensored with the spin bundle must be an honest bundle (with the triple overlap condition satisfied) to accomidate worldvolume fermions. Returning to zero-coupling, if possible, one may then read the charges off of the K-theory class of the gauge bundle on the secondary instanton.

This new sequence appears to classify RR and NS charged configurations, modulo higher differentials. However we have no geometric interpretation for what this sequence computes. In particular, we do not know if this sequence gives the associated graded algebra of some variant of K-theory and thus, after solving some extension problem, provides us with a classification of all states in IIB string theory in terms of a mysterious collection of bundles.

Notice that we included the D7-brane but not any possible S-duals of this brane. This is because we do not know if or how S-duality should act on a system with 7-branes. This may be a deficiency in our proposal. However, in spaces with "no compact directions" 7-brane excitations have an infinite energy backreaction on distant geometry. As a result, it is possible that instantonic 7-brane processes will be infinitely suppressed and therefore in such a limit we may be able to ignore them.

The MMS interpretation of the Atiyah-Hirzebruch spectral sequence has allowed us to blindly calculate equivalence classes of stable configurations modulo dynamical processes. In particular we can calculate which fundamental string and D3-brane states are unstable due to dynamical processes in IIB. We were also able, with the help of a generalized Freed-Witten anomaly, to learn which NS5-brane configurations are unstable. In M-theory we did the same for M2 and M5-branes.

However we are unable to provide more than reckless speculations as to what these equivalence classes may mean. Although we are able to generalize the Atiyah-Hirzebruch spectral sequence, we do not know what mathematical object the new sequence approximates. Perhaps the most promising hint lies in the mysterious E_8 gauge bundle formalism for M-theory, as all of the fields of IIB are those of M-theory compactified on a torus and so the answers in IIB are likely to be those in M-theory with the extra complications arising from this compactification.

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